Indian Statistical Institute
M. Math II year
Number Theory
November 24, 2018

Final Sem exam Time : 3 hours 50 points

1. [5 points] Let p be an odd prime such that $p \equiv 3\ (mod\ 8)$ and $q = \frac{p-1}{2}$ is also prime. Let $n \geq 1$ be smallest number such that $2^n \equiv 1\ (mod\ p)$. Prove that $n = p - 1$.

2. Let $p \equiv 1\ (mod\ 4)$. Then prove that

   (a.)[3 points] $x^2 \equiv -1\ (mod\ p)$ has solutions.

   (b.)[4 points] Use (a) and the method of continued fractions to prove that p is a sum of two squares.

3. [6 points] Find a Pythagorean triangle such that difference of two sides is 1 and every side is atleast 100.

4. [3 points] Find the integral solutions of $x^2 + 2y^2 = 8z + 5$.

5. [5 points] Let ξ be an irrational number. Prove that one of every two consecutive convergents $\frac{h_n}{k_n}$ of ξ satisfies

$$|\xi - \frac{h_n}{k_n}| \leq \frac{1}{2k_n^2}.$$

   (Hint: Use A.M- G.M inequality)

6. [2 points] Show that if a non zero integer n is represented by a binary quadratic form f of discriminant d, then $4an$ is a square modulo $|d|$ where a is the coefficient of x^2 in f.

7. [3 points] Show that binary quadratic form f properly represents an integer n if and only if there is a form equivalent to f in which the coefficient of x^2 is n.

8. [6 points] Prove that an odd prime p is represented by the form $x^2 + 5y^2$ if and only if $p = 5$ or $p \equiv 1$ or $9\ (mod\ 20)$ and that an odd prime p is represented by the form $2x^2 + 2xy + 3y^2$ if and only if $p = 2$ or $p \equiv 3$ or $7\ (mod\ 20)$.

9. [5 points] Prove that there exists a constant A such that

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + A + O\left(\frac{1}{\log x}\right)$$

10. [3 points] Let $\mu(n)$ be the Mӧbius function and

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \\ 0 & \text{else.} \end{cases}$$

   Then prove that

$$-\mu(n)\log n = \sum_{d|n} \mu(d)\Lambda\left(\frac{n}{d}\right).$$

11. [5 points] Let $\psi(x) = \sum_{n \leq x} \Lambda(n)$ and $\theta(x) = \sum_{p \leq x} \log p$. Prove that

$$\lim_{x \longrightarrow \infty} \left(\frac{\psi(x)}{x} - \frac{\theta(x)}{x}\right) = 0.$$